Algebraic Number Theory
End-semestral test

INSTRUCTIONS: Total time 3 hours. Solve any **five** problems. Total maximum marks = 50, all problems carry equal weight. Minkowski's constant for a number field K of degree n is $\frac{n!}{n^n}(\frac{4}{\pi})^s$, where $2s$ is the number of non-real embeddings of K into $\mathbb{C}$. You may use any result proved in the class without proof.

1. True or False (Explain): Let K be a number field and n be a positive integer. Then there are only finitely many ideals in $\mathcal{O}_K$ with norm n.
2. Prove that the ideal $(2, 1+\sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}]$ is not principal.
3. Prove that $\mathbb{Z}[\zeta_5]$ is a PID, where ζ_5 is a primitive fifth root of unity in $\mathbb{C}$.
4. Prove that for any positive integer $m > 1$, and k relatively prime to m, $1 + \omega + \omega^2 + \cdots + \omega^{k-1}$ is a unit in $\mathbb{Z}[\omega]$, where ω is a primitive mth root of unity in $\mathbb{C}$.
5. Let K be a number field having odd degree over $\mathbb{Q}$. Determine the torsion part of the unit group of K.
6. Let K be a real quadratic number field (i.e. has only real embeddings). Prove that there is a unit $u \in \mathcal{O}_K^\times$ such that $\mathcal{O}_K^\times = \{\pm u^k | k \in \mathbb{Z}\}$. Prove that we can choose u with $u > 1$ and that such u is unique (such a generator for $\mathcal{O}_K^\times$ is called the fundamental unit for K).
7. Let K and L be number fields, L/K a Galois extension with Galois group G. Let P be a nonzero prime ideal in $\mathcal{O}_K$ that is inert in L, i.e. $P\mathcal{O}_L$ is a prime ideal. Prove that G is cyclic.